MTHS24 – Exercise sheet 6

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Lecture material

Discussed topics:

- Lippmann-Schwinger equation, Bethe-Salpeter equation, and K-matrix
- Lineshape analysis and Briet-Wigner formula
- Complex algebra, dispersion relations
- Analytic continuation and pole search
- Khury-Treiman equations

References:

- A.D. Martin, T.D. Spearman, Elementary Particle Theory, inSpire
- Review on Novel approaches in hadron spectroscopy by JPAC, inspire

Exercices

6.1 Escape room in the complex plane

(a) Characterize the complex structure of functions $\sqrt{x}$ and $\log(-x)$ by finding the branch points, branch cuts and number of complex (Riemann) sheets in the complex plane.

(b) Repeat (a) for a function $f(x) = \sqrt{x} - \sqrt{x-1}$.

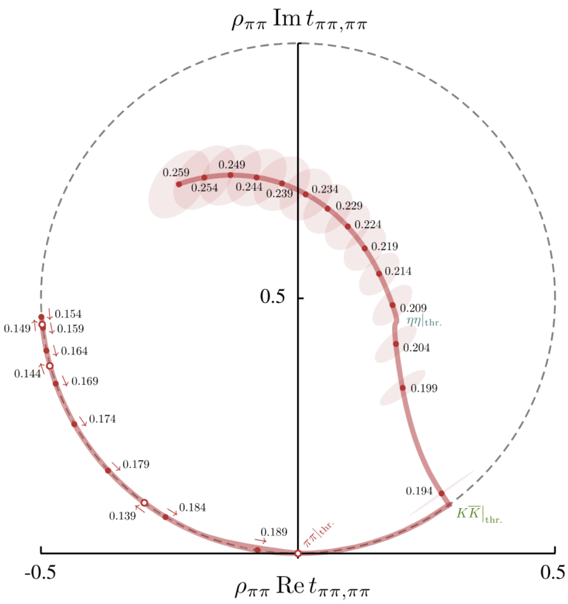
(c) Construct a complex function with two branch points at +i and -i connected by a branch cut.

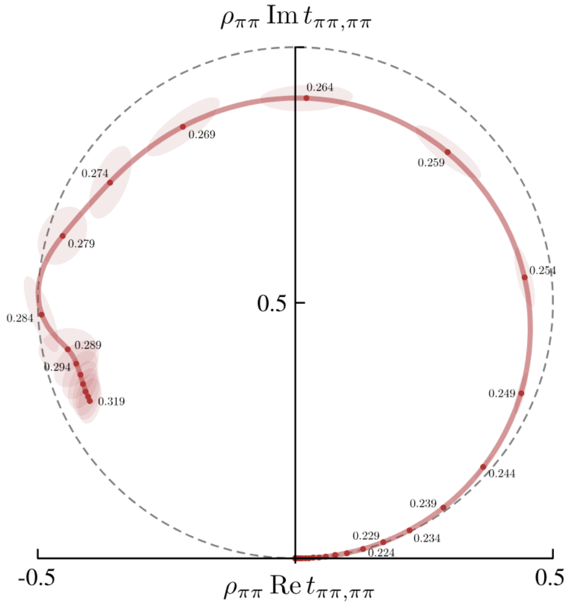
(d) Locate zeros of the function $g(z) = \sqrt{z} + i + 1$.

(e) Find residue of the function $1/g(z)$ by computing a circular integral about the complex pole.

6.2 Argand diagrams from lattice

The $\pi\pi$ scattering with unphysical pion mass ($m_\pi = 391\,\text{MeV}$) for S (left) and D (right) partial waves is studied using lattice calculations. Scattering amplitudes are presented on the Argand diagrams (parametric plot of energy in Real/Imaginary coordinates) as a function of energy of the system. The value are given units of $E_{\text{cm}} \cdot t$ where $t \cdot m_\pi$ = 0.06906.





Using information on the diagrams, answer the following questions:

(a) Estimate masses of K and η particles.

(b) Find the elastic energy region for the S and D waves.

> **Solution:** Elastic region is defined as the range of energy values for which $\pi\pi \to \pi\pi$ process dominates. For the S-wave, the elastic region lies for the $E_{\text{cm}}t$ range of [0.139,0.189] (after this point, the amplitude hits the $K\bar{K}$ threshold. Also, after this point, the curve starts going inside the unitarity circle). For the D-wave, this region exists until value of 0.229.

(c) Locate the energy value for which the S-wave peak.

> **Solution:** The S-wave peaks at the point $E_{\text{cm}}t$ = 0.154, which is at energy $E_{\text{cm}} = 0.872$ GeV.

(d) Estimate the mass and decay width for the D wave resonance.

> **Solution:** The D-wave resonance is observed at $E_{\text{cm}}t = 0.284$, this is the point where there is a kink in the argand diagram. *Note : I can locate where the resonance is. I am confused about how to proceed from there, because I can get center of mass energy from the point, and maybe equate it to the pole value. But I would expect a complex output but I cannot read it out properly from the Argand diagram.*

(e) Sketch the amplitude phase versus energy of the system for both partial waves.

6.3 Cartesian tensors

Cartesian tensors are tensors in three-dimensional Euclidean space. The available tensors are:

- Rank 0: 1
- Rank 1: k^i
- Rank 2: k^ik^j, $delta^{ij}$, $\epsilon^{ijl}k_l$

- Rank 3: ϵ^{ijl}
- Rank 4: combinations of all the above

(a) Show that $\epsilon^{ijl}k_j k_l$ is not a rank 1 tensor.

Solution: The Levi-Civita tensor ϵ^{ijl} is antisymmetric under the exchange of any two indices, i.e. $\epsilon^{ijl} = -\epsilon^{ilj}$. Then

$$\epsilon^{ijl}k_j k_l = -\epsilon^{ilj}k_l k_j$$

and since the product of two components of the same vector is commutative thus $k_j k_l = k_l k_j$ we have:

$$\epsilon^{ijl}k_j k_l = -\epsilon^{ilj}k_j k_l$$

This implies:

$$\epsilon^{ijl}k_j k_l = 0$$

Therefore, $\epsilon^{ijl}k_j k_l$ is identically zero and does not form a valid rank 1 tensor (does not behave as rank 1 tensor).

(b) Show that $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, using the following property: $(\mathbf{A} \times \mathbf{B})^i = \epsilon^{ijl} A_j B_i$
Hint:

$$\epsilon_{ijl}\epsilon_{ij'l'} = \delta_{jj'}\delta_{ll'} - \delta_{jl'}\delta_{j'l} \tag{1}$$

Solution: Starting with the vector identity:

$$(\mathbf{A} \times \mathbf{B})^i = \epsilon^{ijl} A_j B_l$$

Let $\mathbf{C} = \nabla \times \mathbf{A}$, then:

$$C^i = (\nabla \times \mathbf{A})^i = \epsilon^{ijl}\partial_j A_l$$

Now, consider the curl of $\mathbf{C}$, and use the property of the Levi-Civita symbol:

$$\begin{aligned}
(\nabla \times \mathbf{C})^i &= \epsilon^{imn}\partial_m C_n = \epsilon^{imn}\partial_m(\epsilon^{njl}\partial_j A_l) \\
&= \epsilon^{imn}\epsilon^{njl}\partial_m \partial_j A_l \\
&= (\delta^{ij}\delta^{ml} - \delta^{il}\delta^{mj})\partial_m \partial_j A_l \\
&= \partial_i \partial_l A_l - \partial_j \partial_j A_i \\
&= \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}
\end{aligned}$$

(c) Given $\int d^3k\, f(k)\, k^i = 0$ for a scalar function $f(k)$. Show that:

$$\int d^3k\, f(k)\, k^i k^j = \frac{1}{3}\delta^{ij}\int d^3k\, f(k)\, k^2 \,. \tag{2}$$

Solution: By symmetry, the integral $\int d^3k\, f(k)\, k^i k^j$ must be proportional to δ^{ij} because the left-hand side is a rank 2 tensor and the only isotropic rank 2 tensor is proportional to δ^{ij}, then:

$$\int d^3k\, f(k)\, k^i k^j = A\delta^{ij}$$

To find A, take the trace:

$$\int d^3k\, f(k)\, k^i k^i = A\delta^{ii}$$

Since $\delta^{ii} = 3$, we have:

$$\int d^3k\, f(k)\, k^2 = 3A$$

So,

$$A = \frac{1}{3}\int d^3k\, f(k)\, k^2$$

Thus,

$$\int d^3k\, f(k)\, k^i k^j = \frac{1}{3}\delta^{ij}\int d^3k\, f(k)\, k^2$$

(d) Show that

$$\int d^3k\, f(k, \hat{p})\, k^i k^j = \frac{1}{2}\int d^3k\, f\left[k^2 - (k\cdot\hat{p})^2\right]\delta^{ij} + \frac{1}{2}\int d^3k\, f\left[3(k\cdot\hat{p})^2 - k^2\right]\hat{p}^i\hat{p}^j\,. \qquad (3)$$

Solution: The given integral can be split into parts proportional to δ^{ij} and $\hat{p}^i\hat{p}^j$:

$$\int d^3k\, f(k,\hat{p})\, k^i k^j = A\delta^{ij} + B\hat{p}^i\hat{p}^j$$

To find A and B, we take the trace and the contraction with $\hat{p}^i\hat{p}^j$:

(a) Trace:

$$\delta^{ij} \int d^3k\, f(k,\hat{p})\, k^i k^j = A\delta^{ii} + B(\hat{p}^i\hat{p}^i)$$

$$\to \int d^3k\, f(k,\hat{p})\, k^2 = 3A + B$$

(b) Contraction with $\hat{p}^i\hat{p}^j$:

$$\hat{p}^i\hat{p}^j \int d^3k\, f(k,\hat{p})\, k^i k^j = A(\hat{p}\cdot\hat{p}) + B(\hat{p}^i\hat{p}^j\hat{p}^i\hat{p}^j)$$

$$\to \int d^3k\, f(k,\hat{p})\, (k\cdot\hat{p})^2 = A + B$$

(c) Solving for A and B:

$$3A + B = \int d^3k\, f(k,\hat{p})\, k^2$$
$$A + B = \int d^3k\, f(k,\hat{p})\, (k\cdot\hat{p})^2$$

(d) Subtract the second equation from the first:

$$2A = \int d^3k\, f(k,\hat{p})\, (k^2 - (k\cdot\hat{p})^2)$$
$$\text{Thus} \quad A = \frac{1}{2}\int d^3k\, f(k,\hat{p})\, (k^2 - (k\cdot\hat{p})^2)$$

(e) Using A in the second equation:

$$B = \int d^3k\, f(k,\hat{p})\, (k\cdot\hat{p})^2 - A$$
$$B = \int d^3k\, f(k,\hat{p})\, (k\cdot\hat{p})^2 - \frac{1}{2}\int d^3k\, f(k,\hat{p})\, (k^2 - (k\cdot\hat{p})^2)$$
$$\text{So} \quad B = \frac{1}{2}\int d^3k\, f(k,\hat{p})\, (3(k\cdot\hat{p})^2 - k^2)$$

Therefore,

$$\int d^3k\, f(k,\hat{p})\, k^i k^j = \frac{1}{2}\int d^3k\, f\left[k^2 - (k\cdot\hat{p})^2\right]\delta^{ij} + \frac{1}{2}\int d^3k\, f\left[3(k\cdot\hat{p})^2 - k^2\right]\hat{p}^i\hat{p}^j$$

6.4 Photons in Cartesian Basis

Consider photons (or gluons) in Cartesian basis so $a^+_{k_i} = \sum_\lambda a^+_{k_\lambda}\epsilon^i(k_\lambda)$.

(a) Show that the "scalar photon ball" is just a scalar state $\gamma\gamma$ that can be written as:

$$|\gamma\gamma; 0^+\rangle \propto \int d^3k\, \phi(k)\, a^+_{k_i} a^+_{-k_i}|0\rangle\,, \tag{4}$$

where $\phi(k)$ is the momentum wave function.

Solution: Apply the parity operator P:

$$\int d^3k\,\phi(k)\,Pa^+_{k_i}P^+Pa^+_{k_i}P^+|0\rangle = \int d^3k\,\phi(k)\,(-a^+_{-k_i})(-a^+_{k_i})|0\rangle$$

Now take $k \to -k$ then:

$$\int d^3k\,\phi(-k)\,(-a^+_{k_i})(-a^+_{-k_i})|0\rangle = (+)\int d^3k\,\phi(k)\,(a^+_{k_i})a^+_{-k_i})|0\rangle$$

since $\phi(k)$ is a symmetric function (scalar), $\phi(k) = \phi(-k)$. Thus, we arrived to the desired state $|\gamma\gamma; 0^+\rangle$ which is invariant under parity as expected for scalar states.

(b) Show that:

$$|\gamma\gamma; 0^-\rangle \propto \int d^3k\,\phi(k)\,\epsilon_{ijl}k^l a^+_{k_i}a^+_{-k_j}|0\rangle\,. \tag{5}$$

Solution: Apply the parity operator P:

$$\int d^3k\,\phi(k)\,\epsilon_{ijl}k^l Pa^+_{k_i}P^+Pa^+_{-k_j}P^+|0\rangle = \int d^3k\,\phi(k)\,\epsilon_{ijl}k^l(-a^+_{-k_i})(-a^+_{k_j})|0\rangle$$

(a) First way: we take $k \to -k$ then

$$\begin{aligned}\int d^3k\,\phi(k)\,\epsilon_{ijl}k^l Pa^+_{k_i}P^+Pa^+_{-k_j}P^+|0\rangle &= \int d^3k\,\phi(k)\,\epsilon_{ijl}(-k^l)(a^+_{k_i})(a^+_{-k_j})|0\rangle \\ &= -\int d^3k\,\phi(k)\,\epsilon_{ijl}k^l(a^+_{k_i})(a^+_{-k_j})|0\rangle\end{aligned}$$

(b) Second way: Use the antisymmetric property of ϵ_{ijl} i.e. (flipping $i \leftrightarrow j$) where:

$$\epsilon_{ijl}k^l(a^+_{-k_i})(a^+_{k_j}) = -\epsilon_{ijl}k^l(a^+_{k_i})(a^+_{-k_j})$$

Thus, we arrived to the desired state: $|\gamma\gamma; 0^-\rangle$ which is also invariant under parity.

(c) Prove the Lee-Yang theorem which states that one cannot construct a $J = 1$, $\gamma\gamma$ state.

Solution: We seek a rank 1 Cartesian tensor then:

$$|\gamma\gamma; J=1\,; l\rangle = \int d^3k\,\phi_{lij}(k)\,a^+_{k_i}a^+_{-k_j}|0\rangle\,,$$

We must have:

$$\phi_{lij}(k) = \phi(k)t_{lij} + \chi(k)\delta_{ij}k^l\,,$$

then

$$\begin{aligned}
|\gamma\gamma; J=1\,; l\rangle &= \int d^3k\,[\phi(k)t_{lij} + \chi(k)\delta_{ij}k^l]\,a^+_{k_i}a^+_{-k_j}|0\rangle \\
k \to -k &= \int d^3k\,[\phi(-k)t_{l_{ij}} + \chi(-k)\delta_{ij}(-k^l)]\,a^+_{-k_i}a^+_{k_j}|0\rangle \\
&= \int d^3k\,[-\phi(k)t_{lij} - \chi(k)\delta_{ij}k^l]\,a^+_{-k_i}a^+_{k_j}|0\rangle \\
&= -\int d^3k\,[\phi(k)t_{lij} + \chi(k)\delta_{ij}k^l]\,a^+_{k_i}a^+_{-k_j}|0\rangle \\
&= -|\gamma\gamma; J=1\,; l\rangle
\end{aligned}$$

Therefore $|\gamma\gamma; J=1\,; l\rangle = 0$, which validates Lee-Yang theorem that we cannot construct a $J=1$ $\gamma\gamma$ state.
Remark: Note that here $\phi(-k) = -\phi(k)$ since it is not scalar in this case, $\chi(-k) = \chi(k)$ (scalar), and $a^+_{-k_i}a^+_{k_j} = a^+_{k_i}a^+_{-k_j}$ since the creation operators commutes in the case of Bosons (photons).

(d) Show that:

$$|\gamma\gamma\gamma; 0^-\rangle = \int d^3k_1d^3k_2d^3k_3\,\phi(k_1k_2k_3)\,\epsilon_{i_1i_2i_3}\delta(k_1k_2k_3)a^+_{k_1i_1}a^+_{k_2i_2}a^+_{k_3i_3}|0\rangle\,, \tag{6}$$

is a viable state.

Solution: Apply the parity operator P:

$$\begin{aligned}
&\int d^3k_1d^3k_2d^3k_3\,\phi(k_1k_2k_3)\,\epsilon_{i_1i_2i_3}\delta(k_1+k_2+k_3)Pa^+_{k_1i_1}P^+Pa^+_{k_2i_2}P^+Pa^+_{k_3i_3}P^+|0\rangle \\
&= -\int d^3k_1d^3k_2d^3k_3\,\phi(k_1k_2k_3)\,\epsilon_{i_1i_2i_3}\delta(k_1+k_2+k_3)a^+_{-k_1i_1}a^+_{-k_2i_2}a^+_{-k_3i_3}|0\rangle \\
k \to -k &= -\int d^3k_1d^3k_2d^3k_3\,\phi\left((-k_1)(-k_2)(-k_3)\right)\,\epsilon_{i_1i_2i_3}\delta(-k_1-k_2-k_3)a^+_{k_1i_1}a^+_{k_2i_2}a^+_{k_3i_3})|0\rangle \\
&= -\int d^3k_1d^3k_2d^3k_3\,\phi(k_1k_2k_3)\,\epsilon_{i_1i_2i_3}\delta(k_1k_2k_3)a^+_{k_1i_1}a^+_{k_2i_2}a^+_{k_3i_3}|0\rangle
\end{aligned}$$

Thus, this state is a valid state.

(e) Can we construct a $|\gamma\gamma\gamma; 1^-\rangle$ state? (**Hint:** $Pa^+_{k_i}P^+ = -a^+_{-k_i}$)

Solution: We seek to combine 3 vectors to form a vector state. Consider the angular momentum of three photons.

- Combining Angular Momenta:
  - Start by combining two vectors to form an intermediate state.
  - In the case of three vectors $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$, we can form the combinations:

$$(\mathbf{k}_1 \times \mathbf{k}_2)_s = (\mathbf{k}_1\mathbf{k}_2) + (\mathbf{k}_2\mathbf{k}_1)$$

  - This is symmetric in space.
- Cross Product for Antisymmetry:
  - To obtain a 1^- state, we need an antisymmetric combination.
  - Take the cross product of the intermediate state with the third vector $\mathbf{k}_3$:$(\mathbf{k}_1 \times (\mathbf{k}_2 \times \mathbf{k}_3))$
- Constructing the State:
  -Combine all three vectors in such a way that respects the antisymmetry needed for a 1^- state.

$$\int d^3k_1\, d^3k_2\, d^3k_3\, \phi(k_1k_2k_3)\, \epsilon_{i_1i_2i_3}\delta(k_1+k_2+k_3)a^+_{k_1i_1}a^+_{k_2i_2}a^+_{k_3i_3}|0\rangle$$

- Symmetry Considerations:
  - This state is symmetric under permutations of the three momenta $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$.
  - The use of $\epsilon_{i_1i_2i_3}$ ensures the antisymmetry necessary for a pseudoscalar state.
- Parity Check:
  - Under parity, the creation operator transforms as

$$Pa^+_{k_i}P^+ = -a^+_{-k_i}$$

  . - The state under parity transforms as:

$$P|\gamma\gamma\gamma;1^-\rangle = (-1)^3|\gamma\gamma\gamma;1^-\rangle = -|\gamma\gamma\gamma;1^-\rangle$$

  (same as part(d))

This confirms that possibility of constructing a $|\gamma\gamma\gamma;1^-\rangle$ state using the above considerations.