Ruhr-University Bochum Modern Techniques in Hadron Physics

MTHS24 – Exercise sheet 8

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Lecture material

Discussed topics:
- Regge theory
- High energy scattering
- Complex angular momentum
- Unitarity

References:
- P.D.B. Collins, "An Introduction to Regge Theory and High Energy Physics", Inspire
- V.N. Gribov, ("Blue book") "The theory of complex angular momentum", Inspire
- V.N. Gribov, ("Gold book") "Strong interactions of hadrons at high energies", Inspire
- D. Sivers & J. Yellin, "Review of recent work on narrow resonance models" Inspire

Exercices

8.1 Unitarity and Reggeons

Van Hove proposed a physically intuitive picture of a Reggeon by relating it to Feynman diagrams in the cross-channels. We will explore this picture of Reggeization with a simple model.

(a) **Elementary t-channel exchanges**
Consider the amplitude corresponding to a particle with spin-J and mass m_J exchanged in the t-channel as:

$$A^J(s,t) = i\, g_J\, (q_1^{\mu_1}\dots q_1^{\mu_J})\, \frac{P^J_{\mu_1\dots\mu_J,\nu_1\dots\nu_J}(k)}{m_J^2 - t}\, (q_{\bar{2}}^{\nu_1}\dots q_{\bar{2}}^{\nu_J}) \tag{1}$$

where g_J is a coupling constant with dimension $2-2J$ (i.e., $A^J(s,t)$ is dimensionless) and the projector of spin-J is defined from the polarization tensor of rank-$J \geq 1$ as

$$P^J_{\mu_1\dots\mu_J,\nu_1\dots\nu_J}(k) = \frac{(J+1)}{2} \sum_\lambda \epsilon^{\mu_1\dots\mu_J}(k,\lambda)\, \epsilon^{*\nu_1\dots\nu_J}(k,\lambda)\ . \tag{2}$$

Using the exchange momentum $k = q_1 + q_{\bar{3}} = q_1 - q_3$, calculate the amplitudes corresponding to $J = 0, 1, 2$ exchanges in terms of $t = k^2$, the modulus of 3-momentum and cosine of scattering angle in the t-channel frame, q_t and $\cos\theta_t$ respectively. Use the explicit forms of the projectors:

$$P^0(k^2) = 1 \tag{3}$$

$$P^1_{\mu\nu}(k^2) \equiv \tilde{g}_{\mu\nu} = \frac{k_\mu\, k_\nu}{k^2} - g_{\mu\nu} \tag{4}$$

$$P^2_{\mu\nu\alpha\beta}(k^2) = \frac{3}{4}\,(\tilde{g}_{\mu\alpha}\,\tilde{g}_{\nu\beta} + \tilde{g}_{\mu\beta}\,\tilde{g}_{\nu\alpha}) - \frac{1}{2}\,\tilde{g}_{\mu\nu}\,\tilde{g}_{\alpha\beta}\ , \tag{5}$$

and conjecture a generalization of the amplitude for arbitrary integer J.

Hint: Show that in the t-channel frame, the exchange particle is at rest and therefore $\tilde{g}_{\mu\nu}$ reduces to a δ_{ij} with respect to only spacial momenta.

(b) **Unitarity vs Elementary exchanges**
Express the amplitude entirely in terms of invariants s and t. Use the optical theorem to relate the elastic amplitude to a total hadronic cross section:

$$\sigma_{\text{tot}} = \frac{1}{2q\sqrt{s}} \Im A^J(s, t = 0) \ . \tag{6}$$

Unitarity (via the Froissart-Martin bound) prohibits σ_{tot} from growing faster than $\log^2 s$ as $s \to \infty$. What is then the maximal spin a single elementary exchange can have while satisfying this bound? Why is this a problem?

(c) **Van Hove Reggeon**
Consider an amplitude of the form

$$A(s,t) = \sum_{J=0}^{\infty} g\, r^{2J} \frac{(q_t^2 \cos\theta_t)^J}{J - \alpha(t)} \ . \tag{7}$$

Here $\alpha(t) = \alpha(0) + \alpha' t$ is a real, linear Regge trajectory, g is a dimensionless coupling constant and $r \sim 1$ fm is a range parameter. Compare Eq. 7 with Eq. 1, write the mass of the Jth pole, m_J^2, as a function of the Regge parameters $\alpha(0)$ and α'. Interpret the pole structure in terms of the spectrum of particles in the model.

If the sum is truncated to a finite J_{max}, and we take the $s \to \infty$ limit, what is the high energy behavior of the amplitude?

(d) **Analytic continuation in J**
Show that if the summation is kept infinite, the amplitude can be re-summed to something that is entirely analytic in s, t, u, and J.

Hint: Use the Mellin transform

$$\frac{1}{J - \alpha(t)} = \int_0^1 dx\, x^{J - \alpha(t) - 1} \ , \tag{8}$$

to express the amplitude in terms of the Gaussian hypergeometric function and the Euler Beta function

$$B(b, c-b)\, {}_2F_1(1, b, c; z) = \int_0^1 dx\, \frac{x^{b-1}\,(1-x)^{c-b-1}}{1 - x\,z} \ . \tag{9}$$

(e) **Unitarity vs Reggeized exchanges**
Revisit $\mathbf{b}$) with the resummed amplitude. Take the $s \to \infty$ limit and set a limit on the maximal intercept $\alpha(0)$ which is allowed by unitarity.

Hint: Assume that $\alpha(0) > -1$ and use the asymptotic behavior of the hypergeometric function given by

$${}_2F_1(1, b, c; z) \to \frac{\Gamma(c)\,\Gamma(1-b)}{\Gamma(1)\,\Gamma(c-b)}\,(-z)^{-b} \ . \tag{10}$$

(f) **The Reggeon "propagator"**
Modify Eq. 1 to have a definite signature by defining

$$A^{\pm}(s,t) = \frac{1}{2}\left[A(s,t) \pm A(u,t)\right] \ . \tag{11}$$

Repeat **d)** and **e)** with this signatured amplitude. Compare with the canonical form of the Reggeon exchange:

$$A^{\pm}_{\mathbb{R}}(s,t) = \beta(t)\,\frac{1}{2}\left[\pm 1 + e^{-i\pi\alpha(t)}\right]\,\Gamma(-\alpha(t))\,\left(\frac{s}{s_0}\right)^{\alpha(t)} \ . \tag{12}$$

Identify the Regge residue $\beta(t)$ and characteristic scale s_0 in terms of the parameters g_0 and r.

8.2 Veneziano Amplitude

The quintessential dual amplitude was first proposed by Veneziano for $\omega \to 3\pi$ and later applied to elastic $\pi\pi$ scattering by Shapiro and Lovelace. Consider the $\pi^+\pi^-$ scattering amplitude of the form

$$\mathcal{A}(s,t,u) = V(s,t) + V(s,u) - V(t,u) \ . \tag{13}$$

with each

$$V(s,t) = \frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))} \ , \tag{14}$$

where $\alpha(s) = \alpha(0) + \alpha' s$ is a real, linear Regge trajectory with $\alpha' > 0$.

(a) **Duality**
Show that the function $V(s,t)$ is symmetric in $s \leftrightarrow t$ and dual, i.e., it can be written entirely as a sum of either s-channel poles OR t-channel poles but never both simultaneously. Compare with the Reggeized amplitude in the previous problem, was that amplitude dual?

Hint: Relate $V(s,t)$ to the Euler Beta function

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \tag{15}$$

and use the identities $B(x,y) = B(y,x)$ and

$$B(p-x, q-y) = \sum_{J=1}^{\infty} \frac{\Gamma(J-p+1+x)}{\Gamma(J)\,\Gamma(-p+1+x)}\,\frac{1}{J-1+q-y} \ . \tag{16}$$

(b) **Isospin basis**
Define the s-channel isospin basis through

$$\begin{pmatrix} \mathcal{A}^{(0)}(s,t,u) \\ \mathcal{A}^{(1)}(s,t,u) \\ \mathcal{A}^{(2)}(s,t,u) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} \mathcal{A}(s,t,u) \\ \mathcal{A}(t,s,u) \\ \mathcal{A}(u,t,s) \end{pmatrix} \ . \tag{17}$$

Write down the definite-isospin amplitudes in terms of V's. Comment on the symmetry properties of each isospin amplitude with respect to $t \leftrightarrow u$.

(c) **Chew-Frautshi plot**
Locate where each $\mathcal{A}^{(I)}(s,t,u)$ will have poles in the s-channel physical region. What is their residue? Draw a schematic Chew-Frautschi plot of the resonance spectrum in each isospin channel.

(d) **Regge limit**
Now consider the limit $t \to \infty$ and $u \to -\infty$ with $s \le 0$ is fixed. What is the asymptotic behavior of $V(s,t)$ and $V(s,u)$? Assume that $V(t,u)$ vanishes faster than any power of s in this limit. What is the resulting behavior of the isospin amplitudes $\mathcal{A}^{(I)}(s,t,u)$ in this limit?

Hint: Use the Sterling approximation of the Γ function., i.e. as $|x| \to \infty$

$$\Gamma(x) \to \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x \ . \tag{18}$$

(e) **Ancestors and Strings**
Consider the model now with a complex trajectory $\alpha(s) = a_0 + \alpha' s + i\Gamma$ with $\Gamma > 0$ to move the poles off the real axis. Reexamine the the Chew-Frautshi plot for the $I = 1$ amplitude using this trajectory, why is the resulting spectrum problematic? Try a real but non-linear trajectory, say $\alpha(s) = \alpha_0 + \alpha' s + \alpha'' s^2$, what is the spectrum like now?

Compare the requirements of the trajectory for $V(s,t)$ to make sense with the energy levels of a rotating relativistic string with a string tension T:

$$E_J^2 = \frac{1}{2\pi T} J \; . \tag{19}$$

What is a possible microscopic picture of hadrons if the Veneziano amplitude is believed?

8.3 Sommerfeld-Watson Transform

(a) **Geometric series**
Prove the well known resummation of the geometric series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1 \tag{20}$$

can be analytically continued to $|x| \geq 1$ with the Sommerfeld-Watson Transform.

Assume that $|x| > 1$ and show that the summation can be written as an integral over the complex plane

$$\int \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell} = 1 + x + x^2 + \ldots \; . \tag{21}$$

Draw the contour around which the above integration should be taken (careful with orientations and signs). Deform the contour such that you can relate Eq. 21 to the series

$$\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots = \frac{1}{1 - \frac{1}{x}} \tag{22}$$

and arrive at Eq. 20.

(b) **Van Hove Reggeon**
Revisit the Regge behavior of Eq. 7 using the S-W transform. How does the inclusion of poles at $\alpha(s) = \ell$ change the contour of integration and the leading contribution to the asymptotic behavior?

8.4 Finite Energy Sum Rules

Consider z a complex variable and α a real fixed parameter. What is the analytic structure of the function z^α? What is the discontinuity across the cut?

Write a Cauchy contour C surrounding the cut and closing it with a circle of radius Λ in the complex z plane, and check that

$$\oint_C z^\alpha \mathrm{d}z = 0 \tag{23}$$

You can start with the simple case $\alpha = 1/2$, *i.e.* $\sqrt{z}$, then generalize to any real α.